Extracting the Risk-Neutral Distribution of the Federal Funds Rate from Option Prices

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Abstract

Interest rate futures are among the most liquid derivative contracts and they are traded on a well-developed financial markets segment. By embedding information about the market prices, they provide policy makers and market practitioners with an opportunity to derive the expected part of the future monetary policy actions. Similar to futures contracts, options on interest rate futures are also well established and actively traded and were initially introduced to match the needs of hedgers and risk takers. This paper shows how the risk-neutral distribution of the future path of US interest rates can be derived from option quotes. Monitoring the distribution of the market expectations helps policy makers to analyze how incoming information affects market view not only with respect to future levels but also with respect to probability of future outcomes. The information extracted from option prices can be used for market monitoring and analyses and for decision making and portfolio positioning as well.

1 Introduction

The Fed Funds are typically overnight loans between banks of their deposits at the Federal Reserve. The interest rate in these loans is the federal funds rate and it is a barometer of the tightness of credit market conditions in the banking system (Mishkin 1992). Although the fed funds rate is generically established by the market forces, by using open market operations the Federal Reserve has been trying to achieve the target rate set by the Open Market Committee. Historically, the Federal Reserve has been successful as in the last five years the deviation of the monthly average fed funds rate from its day-weighted average target level has been within 5 basis points (Carlson, Craig, Melick 2005).

Federal funds futures are one month interest rate futures contracts, which are generally the difference between the average market rate during the delivery month and the futures rate at the time the contract was bought, multiplied by the notional amount, and they are settled in cash. Federal funds futures contracts are traded on CBOT and are quoted in terms of a price, which is calculated as 100 minus the realized average fed funds rate for the delivery month. By definition under the no arbitrage condition the discounted market price of a future contract equals the basic asset spot price. In the case of federal funds futures the basic price equals 100 minus the average overnight federal funds rate for the calendar month. As mentioned in Krueger, Kuttner (1996) the average includes weekend and holidays. For example, the quoted price on December 5, 2005 for the March 2006 contract was 97.350, which means that the expected average fed funds rate over March 2006 was 2.650%. Daily prices for overnight federal funds rates are published by the FRBNY for each working day. Last working day prices are carried over during weekends and holidays.

Based on some restrictive assumptions of no-arbitrage condition, constant funding rate and constant risk premium, the fed futures prices will reflect prevailing market expectations of future spot federal funds rate. This relationship is summarized under a common term of rational expectation hypothesis (REH). Krueger, Kuttner (1996) tested unbiasedness and rationality of one- and two-month fed funds futures contracts for the period 1989-1994. They conclude that fed funds prices are biased but rational (incorporating all available information) predictor of future fed funds rate due to the existence of a small premium. Robertson and Thornton (1997) also document the existence of positive bias in the fed funds futures forecast, which is attributed to the risk premium. According to the authors, a risk premium exists because of the hedging premium that is paid by the large banks to hedge against increases fed funds rate.

In contrast to the fed funds futures prices, which can be used to extract only expectations of the future level of fed funds rate, the option on the fed funds futures incorporate market expectations on the future price, the implied market volatility and the possible skew of the underlying asset price. By deriving the implied probability distribution over different possible paths of the underlying asset price we will gather a larger set of market information. For example, the derivation of probability distribution can lead to the estimation of a probability for the realization of tail event. The skewness of the distribution will give us the direction of change in price of the fed funds future. This information can be used for generating different scenarios and different trading strategies, given the relative probability of their realization is known.

2 Methodology

In recent years the idea that option prices could be used to gauge the distribution of the underlying asset prices has been gaining much popularity. Option prices provide a good deal of information about the risk-neutral densities, which can be traced out by observing the entire spectrum of calls and puts available in the market. Unfortunately, a major constrain to this approach is the shortage of data as there are many assets, especially when it boils down to the interest rate products, which are traded within a very constricted window of strike prices.

The attitude of the market toward different risks associated with a particular asset along with the expectations about the future movements of the asset price can be encapsulated by the distribution of the prices of the asset and its derivatives, which will produce the so called risk-neutral probability distribution. Such information can be extracted from the prices of different options on an underlying asset. One could trace out the entire risk-neutral distribution of the underlying asset simply by observing all the available calls and puts, which have the same expiration. To see why this is true, one can think of, for example, the market price of a call option as the risk-neutral expectation of its payoff at maturity discounted to the present. The Black-Scholes model for pricing a European call option when the underlying asset is a futures contract presents this idea:

$$c = e^{-rT}[F_0 \mathbb{P}(F > K) - K\mathbb{Q}(F > K)] \tag{2.1}$$

where r is the risk-free interest rate, F_0 is the price of the future at time 0, K is the strike price and $\mathbb{P}$ is the probability that the futures price is less than the strike at time 0 and $\mathbb{Q}$ is the probability that the futures price is less than the strike at time T. This can be written also in the form:

$$c(0, K, T) = e^{-rT}[F_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2)] \tag{2.2}$$

where

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln \frac{F_0}{K} - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

and $\mathcal{N}$ is the cummulative distribution function of the standard normal variable, K is the strike price and σ is the volatility of the underlying asset.

The general approach for deriving the probability distribution from option prices, assumes that implied volatility can gives us enough information about the first moments of the volatility smile, which in turn is used to describe the distribution of the future asset prices. Malz (1996) and Shimko (1993) decided to fit a function, which is dependent on the different strike prices, to the implied volatilities. The probability density function, then, is obtained by differentiating the analytical price of the European option twice with respect to the exercise price.

$$\frac{\partial^2 c(0,K,T)}{\partial K^2} = e^{-rT}\pi(K) \tag{2.3}$$

where $\pi(K)$, which depends on the strike price K, is the PDF of the underlying asset.

A different approach is undertaken by Melick and Thomas (1997) and Bahra (1996), which assume that the risk-neutral distribution is a mixture of univariate log-normal distributions. Bahra (1996) prefers to use a two log-normal mixture, by placing a weight parameter on both distributions, whose weights eventually sum to one. To estimate the parameters of the two distributions along with the weight parameter, Bahra (1996) uses a procedure, which minimizes the sum of the squared errors, with respect to the distribution parameters.

Soderlind and Svensson (1997) presented an extension of this approach. Investigating the distribution of bond option prices, they show that the risk-neutral distribution can be generated by a distribution of log stochastic discount factor and the log bond price, which in turn is a mixture of bivariate normal distributions.

A relatively simpler approach proposed by Carlson, Craig and Melick (2005) could be employed to recover the PDF using OLS framework. Apart from its tractability, another advantage of this method,is that it is possible to estimate the probabilities with several possible paths for the evolution of price of the underlying asset.

3 Model Description

In what will follow we will investigate how the model framework presented above can be applied to the derivation of the distribution of the interest rate given by the futures contracts on the federal funds. We will use the model presented in Malz(1997). The model rests on the observation that not all options with on the same underlying price have the same volatility but rather the volatility is a function of the exercise price. Normally, out-of-the-money options and deep in-the-money options have volatilities higher than at-the-money options. This phonemonon constitutes the so-called "volatility smile", which is an evidence that Black-Scholes model does not hold exactly. So, instead of a constant volatility, we have to use an implied volatility function $\sigma(K)$ that is dependent on the exercise price. There are several things that should be noted about the options which are traded in the market:

- Out-of-the-money options normally have higher implied volatilities than at-the-money options, which means that the market perceives the distribution of the interest rates to be leptokurtotic (or 'fat-tailed') (Malz 1997).

- Out-of-the-money call options often have implied volatilities that differ from out-of-the-money puts with the same strike price. This means that the market greater probability for an interest rate move in one of the directions, that is, the probabilities of going up or down are not equally likely.

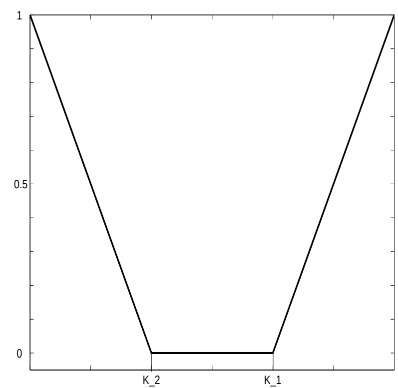

Figure 1: Strangle Payoff

Often options are sold in combinations. One such combination is the *strangle*. This is combination of two out-of-the-money options - a long call and a long put with the same delta (usually 25%)(Figure 1). We should note that the delta of an at-the-money option is close to 0.5 while the delta of a put option is expressed in negative terms. Another combination, which also involves two out-of-the-money options is the *risk reversal*. The risk reversal consists of a long call and a short put (Figure 2). The prices of these combinations are expressed in vols instead of currency units. Normally, the prices of strangles and risk reversals are given as the spread over the at-the-money volatility. It is important to mention that in accordance with what we stated above, strangle spreads would not be zero since out-of-the-money options does not have the same implied volatility as at-the-money options. The latter is true because market participants do not consider interest rate futures to be lognormally distributed but "fat-tailed". Thus, strangles indicate the degree of curvature of the volatility smile (Malz 1997).

We can use an interpolation technique by using a second-order Taylor approximation, as described in Malz (1997), to derive the implied volatility as a function of the option's delta.

$$\sigma_\delta = a_1 ATM_t + a_2 RR_t(\delta - 0.50) + a_3 STR_t(\delta - 0.50)^2 \qquad (3.1)$$

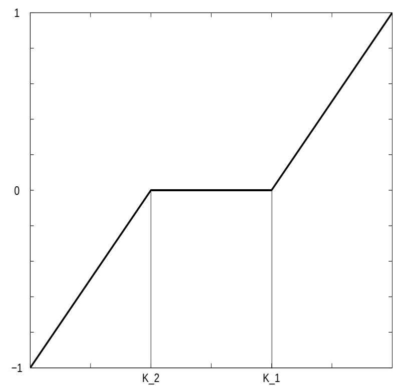

Figure 2: Risk Reversal Payoff

where ATM is the at-the-money volatility, RR - the risk reversal and STR - the strangle at time t and δ is the delta of the option.

To find the coefficients a_1, a_2 and a_3 we need to express the option combinations in terms of 25-delta call and a 25-delta put options on the federal funds futures:

$$
\begin{aligned}
RR &= \sigma_\delta^c - \sigma_\delta^p \qquad (3.2)\\
STR &= \frac{\sigma_\delta^c + \sigma_\delta^p}{2} - ATM \qquad (3.3)
\end{aligned}
$$

where σ_δ^c and σ_δ^p are respectively the volatilities of the call and the put option. Solving this system of equations for σ_δ^c and σ_δ^p we get:

$$
\begin{aligned}
\sigma_\delta^c &= ATM + STR + \frac{RR}{2} \qquad (3.4)\\
\sigma_\delta^p &= ATM + STR - \frac{RR}{2} \qquad (3.5)
\end{aligned}
$$

Having in mind that $\sigma_{-0.25}^p \approx \sigma_{0.75}^c$ and substituting into (3.1) this translates into:

$$
\begin{aligned}
ATM &= \sigma_{0.50} = \sigma_\delta = a_1 ATM_t + a_2 RR_t.0 + a_3 STR_t.0 = a_1 ATM_t\\
RR &= \sigma_{0.25}^c - \sigma_{-0.25}^p = -\frac{a_2}{2} RR\\
STR &= \frac{\sigma_{0.25}^c + \sigma_{-0.25}^p}{2} - ATM = (0.25)^2 a_3 STR
\end{aligned}
$$

The resulting equation for the implied volatility as a function of δ takes the form:

$$\sigma_\delta = ATM_t - 2RR_t(\delta - 0.50) + 16STR_t(\delta - 0.50)^2 \tag{3.6}$$

So far, we have expressed the implied volatility as a function of the delta of the options with the intent to capture some information about the volatility smile. Reviewing the formula above and going back to the definition of the option strategies, we can see that there are several intuitive points behind it. The first term - the at-the-money volatility, gives the "location" of the smile, that is, it anchors the general level of the implied volatility. The second component - the risk reversal price, indicates the skewness of the smile. This provides the level of the directional risk. If there is a positive net premium, the probability for the underlying asset price to go up is higher. As for the strangle price, it provides us with information about the curvature of the smile.

The second step in this approach is to find the implied volatility function as a function of the exercise price, which will give us the level of the volatility corresponding on each exercise price. In order to do that we need to be able to find the delta of an option. By definition, the delta of an option is the sensitivity of the option with respect to the price of the underlying asset or how the price of the option changes when there is a change in price of the asset. Mathematically this is given by:

$$\delta_S(K) = \frac{\partial c(0, K, T)}{\partial S}$$

If we now substitute the expression above into Equation (3.6), we will find how the implied volatility looks like in terms of the strike price K. Since, by assumption, the implied volatility is a function of the strike price if we solve the following equation for σ, we will find the different volatilities that correspond to each strike price, that is the interpolated volatility smile:

$$\sigma_\delta(K) - ATM_t + 2RR_t\left(\frac{\partial c}{\partial S}(K) - 0.50\right) - 16STR_t\left(\frac{\partial c}{\partial S}(K) - 0.50\right)^2 = 0 \tag{3.7}$$

Once we have found the volatility spectrum, the next stage is to use the analytical formula for pricing option contracts to find the whole range of implied option prices, which will be used to construct the distribution of the underlying asset price. The final step is to use Equation (2.3) and differentiate twice with respect to the strike price.

4 Estimation

Before we begin with our estimation, we need to elaborate more on the type of option contacts that will be used. Since the nature of the futures contracts is such that their price cannot exceed 100, there will be strike prices that will be irrelevant and the resulting function will not capture the true distribution of the underlying asset if plain vanilla options are used in the estimation procedure. This fact prompted us to use barrier options instead or more specifically "up-and-out" contracts with a barrier of 100. Another problem is the type of options, which are quoted in the market. They are typically American options, which give the owner the right to exercise them at any time before maturity. Thus, these options have an early exercise premium imbedded in them which distorts to some extent the relationship between the PDF of the underlying asset and the analytical option price (the European option price). In our case, however, since the underlying asset is the fed funds future, we can neglect the early exercise premium because these options are almost never exercised early (Carlson, Craig, Melick 2005).

The analytical formula for the price of the "up-and-out" call option contract written on an interest rate future when $L > K$, where L is the barrier, is given by (Hull 5th ed. 2002):

$$c_{uo} = e^{-rT} \left\{ L \left[\mathcal{N}(-x) - \mathcal{N}(-y) \right] - \frac{KF_0}{L} \left[\mathcal{N}(-x + \sigma\sqrt{T}) - \mathcal{N}(-y + \sigma\sqrt{T}) \right] \right\} \tag{4.1}$$

where

$$x = \frac{\ln \frac{L^2}{F_0 K} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

$$y = \frac{\ln \frac{L}{F_0} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

If $L < K$ then $c_{uo} = 0$. Similarly, the price of a "up-and-out" put option contract written on an interest rate future when $L > K$ is given by:

$$p_{uo} = e^{-rT} \left[K\mathcal{N}(-d_2) - F_0\mathcal{N}(-d_1) + L\mathcal{N}(-x) - \frac{KF_0}{L}\mathcal{N}(-x + \sigma\sqrt{T}) \right] \tag{4.2}$$

An important input in the model is the market quotes for price of the strangle, the risk reversal and at-the-maturity option. The market prices will ensure that we will capture the risk-neutral probability distribution. Unfortunately, we were not able to find real market quotes for the strangle and the risk reversal and thus, we decided to work with synthetic strangles and risk reversals whose prices will be given in vols and will be based on market quotes for call and put options. How exactly these prices are extracted is explained in Appendix II.

Once we have acquired the prices of the options in Equation (3.7), we need to find an explicit representation of the delta of the call option. This is done by differentiating once Equation (4.1) with respect to the price of the future contract F_0. Since the expression for the first derivative is quite complicated, we used symbolic differentiation in MATLAB to carry out the differentiation. After we have found the derivative, a numerical procedure is involved for solving Equation (3.7) for each K. For the purpose, we employed again a MATLAB function and we found a discrete representation of the implied volatility function $\sigma_\delta(K)$.

The final step in extracting the PDF of the fed funds futures is the calculation first of the cumulative distribution function $\Pi(K)$, which gives the probability that the future price of the fed funds future will be less than or equal to K. Consequently, we will calculate the probability density function $\pi(K)$.The derivation of the theoretical PDF can be found in Appendix I. One way to do the differentiation is to find the difference quotients. The estimated cumulative distribution function is given by:

$$\hat{\Pi}(K) = e^{-rT}\left[\frac{c(\sigma_\delta(K), K) - c(\sigma_\delta(K), K - \Delta K)}{\Delta K} + 1\right] \tag{4.3}$$

The estimated probability density function is:

$$\hat{\pi}(K) = \frac{\hat{\Pi}(K) - \hat{\Pi}(K - \Delta K)}{\Delta K} \tag{4.4}$$

where ΔK is the step size. If we repeat for each K we will get the entire density function.

5 Results

The data that we used in our estimation were taken from Bloomberg. We used quotes for the first federal funds future contracts expiring at end of the month. Two probability density functions are shown on Figure 3, which represent the evolvement of the PDF with time. It is evident that the funtion on Oct. 23 2005 is a bit "flatter" than the one on Jan. 5 2006, which is consistent with the general intuition. Since the Federal Reserve increased its target rate to 4.25% early in December, it is reasonable to expect the probability for the interest rate to be closer to 4.50% to increase. This will "extend" the function upwards, narrowing its tails. Another observation is the level of kurtosis and skewness of the density functions. The general assumption under the Black-Scholes model is that the distribution of the fed funds futures price is lognormal. However, the skewness and the kurtosis of

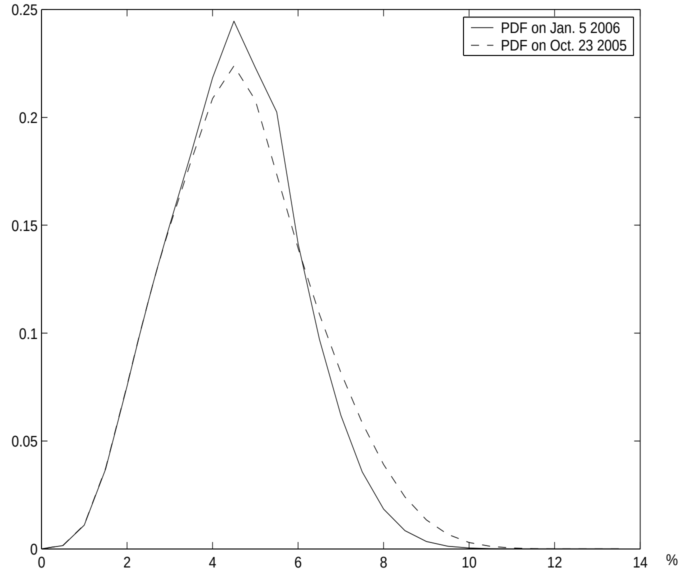

Figure 3: Probability Density Function

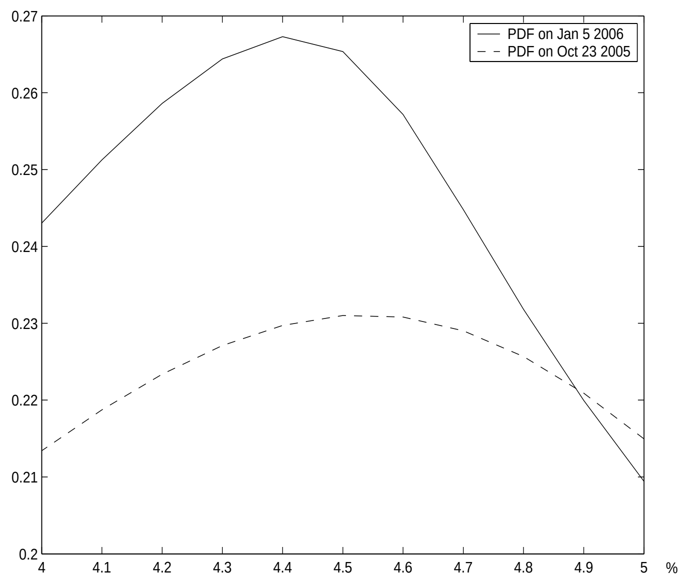

Figure 4: Probability Density Function

the estimated distribution are not exactly the same as those of the lognormal.

The second graph represents the same PDF's but this time it gives only the distribution in a very narrow range. The interval we have chosen corresponds to the available strike prices, which are quoted in Bloomberg. It shows only the top of the curve and it can be seen that relative to the distribution on Oct 23, the one on Jan. 5 is skewed to the left. The interpretation of this fact lies in the observation that on Jan 5 2006 the market gives greater probability for the interest rates to fall in the future. This is consistent with the prevailing market view in the beginning of January 2006 that the Federal Reserve will soon stop the hiking cycle. The kurtosis, on the other hand, reflects the expectation on extreme moves in the futures price. The fatter the tails, the higher the probability of large deviations of the underlying price.

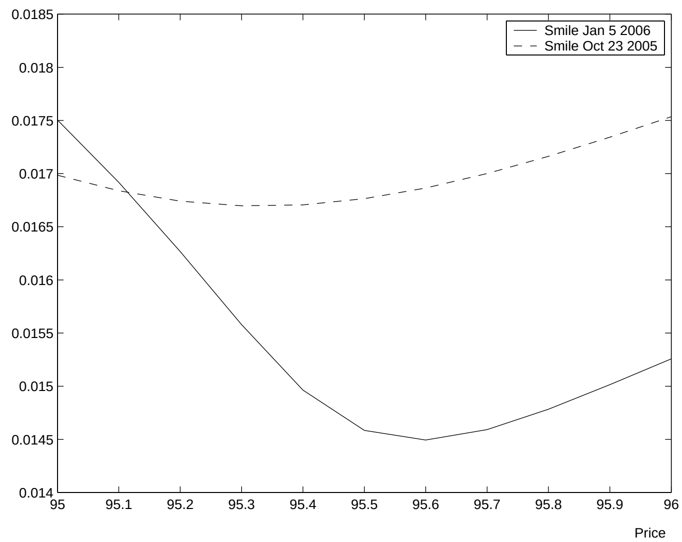

Figure 5: Volatility Smile

The volatility smiles in the Figure 5 confirm the fact that the PDF on Jan. 5 2006 has a greater degree of skewness and kurtosis than the one on Oct. 23 2005. One possible reason for the difference between the two graphs can be purely technical. Since the underlying asset is an one month future, the contracts on Oct. 23 are closer to the exparation date than the ones on Jan. 5 thus, as it can be expected, the closer the exparation date, the lesser the volatility. Furthermore, it is also evident that the overall volatility on Jan. 5 is less than the volatility on Oct. 23 and this can be attributed to the increased confidence among the market participants that the fed funds target rate will soon be increased to 4.50%.

A very important question that should also be addressed is the level of accuracy of the model, which, to a greater extent, depends on whether the interpolated smile represents a good approximation of the true volatility structure. Figure 6 represents the call option implied volatilities as quoted by Bloomberg and the interpolated volatility smile calculated using Equation (3.6). The x-axis shows the delta of all of the available options and as we have noted options with 50-delta call volatility are the at-the-money options whereas 25- and 75-delta calls are respectively in-the-money and out-of-the-money options. These three options capture the position of the smile and as it can be seen from the graph, the fit between the estimated and the actual volatilities is very good. For the less actively traded options, such as the deep in-the-the-money or out-of-the-money options (or, say, 10- and 90-delta options) the implied volatilities set by the dealers have a greater degree of judgement than those for the more actively traded ones (Malz 1997). This fact can explain the larger errors, which are evident around the edges of the smile.

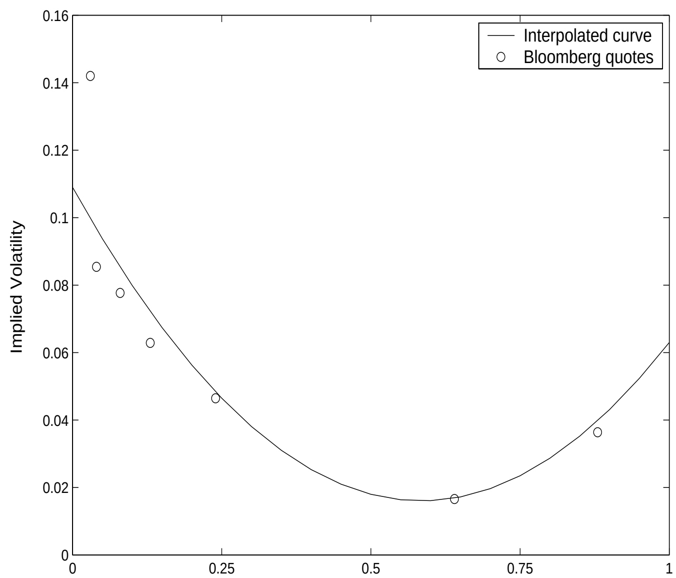

Figure 6: Interpolated Implied Volatility on Jan. 10 2006

Another possible room for error is the estimation of the implied volatilities. Since we are using a numerical method for solving Equation (3.7) there will be some discrepancies between the true implied volatility and our estimate. However, as it can be seen from the graph, the estimated implied volatility approximates reasonably well the real volatility of the options traded in the market.

6 Conclusion

The analysis and the estimations we carried out show that information on future interest rates path can be extracted from option prices. Generally, futures prices on fed funds give indications of the consensus view about the expected prices of the underlying assets. The futures, however, present a static view of the market expectations as they give the anticipated price of the asset at a specific point in time under specific assumptions. In this way part of the market information about the interim dynamics of the basic asset price till that future date is not revealed. Options, on the other hand, show the possible evolution paths of the underlying asset price and hence provide more information about the distribution of possible monetary policy decisions and potential dominance of certain scenarios. This fact helps policy makers to evaluate market perception and possibly change communication in order to influence the market view if departs from their intentions. Investment managers and market analysts are also regular users, because with such a tool they will access market consensus and position accordingly to book profit when consensus changes occur.

Once the probability distibution is extracted, the first moments can be found, which can be used to examine the market view in its completeness. Consequently, the investors can use the information in option prices more effectvely. The higher moments of the federal funds distibution can be used to judge the riskness of a position by placing probabilities to different outcomes. The third moment, the skew of the distribution, can give market participants insights about he perception of the market in what direction the interest rates will move. The fourth moment, the kurtosis, provides signals about how consistent the market opinion is about the future moves of the interest rate across different possible scenarios.

Because of the specifics of the underlined asset (federal funds rate is bounded from below) in this paper the probability distribution is derived based on a model for pricing barrier options. This feature is in contrast with studies on options on exchange rates and asset prices which are based on direct application of B-S model. Possible future development of the model could condition changes in the shape of the probability distribution across different monetary policy scenarios with the flow of new macroeconomic information on the market and changes in Federal Reserve communication.

7 Appendix I Derivation of the PDF

Since we are working with barrier options, we need to redefine the probability density funciton of the payout function. Let $\Phi(S(T))$ be our payout function. By the definition for the "up-and-out" call option contract we have:

$$\Phi^L(S) = \max(S - K)I_{\{S<L\}}$$

where in the case of the fed funds futures $L = 100$ and $I_{\{S<L\}}$ denotes the indicator function. In oder words, this means that the price of the option contract at time of maturity T equals:

$$\begin{cases} \Phi(S(T)) = \max(S(T) - K) & \text{if } S(t) < 100 \text{ for all } t \in [0, T] \\ 0 & \text{if } S(t) \geq 100 \text{ for some } t \in [0, T] \end{cases}$$

Here, we need to point out one important feature of the distribution of the barrier options. Since the underlying asset price of the "up-and-out" call option contract cannot exceed 100 and clearly cannot fall below 0, we have a closed interval over which the whole density is spread out. This gives rise to a stochastic process absorbed at some point that is the barrier, say L. If $F(t)$ is the price process of the fed fund future, we say it has a point mass at $F = L = 100$ (the probability that the process is absorbed before some time t) along with its density. We will denote this as $F_L(t)$. In this case the density of the process has a support on the interval $(-\infty, L)$ when $a < L$. In our case, when $L = 100$ and we have a "up-and-out" call option contract, the intverval over which we have our density is $[0, 100)$. Adopting the standard theory assumption that the price process for the future is a martingale with respect to the forward risk-neutral distribution Q that is its drift coeffiecient is zero, we can write the process for $F(t)$ as:

$$\begin{aligned} dF(x) &= \sigma dW(t) \\ F(0) &= a \end{aligned} \tag{7.1}$$

where σ is the volatility of the price process and W is a standard Wiener process. Furthermore, we can can express the solution of the above equation as:

$$F(T) = e^{\ln a + \sigma W(T)} = e^{X(T)}$$

Using this notation, we can rewrite:

$$F_L(t) = e^{X_{\ln L(t)}}$$

Now, the density function of the absorbed process $X_{\ln L(t)}$ is given by:

$$f_L(x) = \phi(x; \ln a - \frac{\sigma^2 T}{2}, \sigma\sqrt{t}) - \frac{a}{L}\phi(x; \ln \frac{L^2}{a} - \frac{\sigma^2 T}{2}, \sigma\sqrt{t})$$

where, ϕ is the probability density function (Bjork, 1998). Using the standard Black-Scholes framework, we can write the price of the "up-and-out" call option as:

$$\begin{aligned} c(0,K,T) &= e^{-rT}\mathbb{E}_a^Q[\max(F_L(T)-K,0)] \\ &= e^{-rT}\int_{\mathbb{R}} I_{[0,\ln L)}\max(F_L(x;T)-K,0)f_L(x)dx \end{aligned}$$

where the expectation is taken with respect to the risk-neutral distribution Q with an initial value for the futures price process equal to $F(0)=a$. After several transformation, whcih can be found in (Bjork, 1998), we get the following result:

$$\begin{aligned} c(0,K,T) &= e^{-rT}\mathbb{E}^Q[\max(F_L(T)-K,0)] \\ &= e^{-rT}\left(\int_{\mathbb{R}}\max(F_L(x;T)-K,0)\phi(x;\ln a-\frac{\sigma^2T}{2},\sigma\sqrt{t})\right) \\ &\quad -e^{-rT}\left(\frac{a}{L}\int_{\mathbb{R}}\max(F_L(x;T)-K,0)\phi(x;\ln\frac{L^2}{a}-\frac{\sigma^2T}{2},\sigma\sqrt{t})\right) \end{aligned}$$

There is an important implication of this result. As we can see from the last two expectations, the contract function under the integral sign is no longer a barrier option but a standard call option. Moreover, both expectations lead to normal densities and having in mind the fact that the sum of two normal variables is again normal, we can differentiate the pricing function with respect to K in order to find the density function:

$$\frac{\partial c(0,K,T)}{\partial K}=e^{-rT}[(1-\Pi_1(K))-\frac{a}{L}(1-\Pi_2(K))]$$

where $\Pi_1(x)$ and $\Pi_2(x)$ are two the risk neutral normal cummulative distribution functions, and the probability density function is given by:

$$\frac{\partial^2 c(0,K,T)}{\partial K^2}=e^{-rT}[\pi_1(K)-\frac{a}{L}\pi_2(K)]$$

8 Appendix II Extraction of the strangle and risk reverse prices

Normally, the prices of the strangles and risk reversals are quoted in vols rather than actual currencies. Since we were not able to find actual quotes for these option strategies, we found their price synthetically. The prices of a strangle and risk reversal are given respectively by:

$$\begin{aligned} STR_t &= e^{r(T-t)}\mathbb{E}[\max(F_L - K_1, 0) + \max(F_L - K_2, 0)] \\ &= c_{uo}(\sigma) + p_{uo}(\sigma) \end{aligned}$$

and

$$\begin{aligned} RR_t &= e^{r(T-t)}\mathbb{E}[\max(F_L - K_1, 0) - \max(F_L - K_2, 0)] \\ &= c_{uo}(\sigma) - p_{uo}(\sigma) \end{aligned}$$

where $c_{uo}(\sigma)$ and $p_{uo}(\sigma)$ are functions of σ and are given by Equation (4.1) and Equation (4.2) and F_L is the same as described in Appendinx I. The prices of the strangle and the risk reversal at time 0 in terms of dollars are taken from Bloomberg and substituted in the above equations. The equations, then, are solved for σ using a numerical solver in MATLAB and the retrieved values for the implied volatilities are used for finding the solution of Equation (3.7).

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